УДК 664

Interpolation rule for estimating the greatest stress in a structure subject to creep

Aret V.A., Eliseev A.M., Fedorova L.A.
kingalya@mail.ru

Saint-Petersburg National Research University of Information Technologies, Mechanics and Optics.
Institute of Refrigeration and Biotechnology
191002, St. Petersburg, Lomonosov str., 9

Several widely used in Food technologies simple structures made of material for which the creep strain rate is proportional to the nth power of stress are studied with a view to determining the way in which the greatest stress in the structure varies with the exponent n when the geometry of the structure and the load carried by the structure remain unchanged.

***From a study of the results a simple empirical linear rule for interpolating for greatest stress between the specially simple cases n=1 and n→∞ (which correspond to exact analogues of linear elastic and perfectly plastic material respectively) is devised. The rule was verified on the base K-method* [2].**
Keywords: creep, relative stress concentration factor, interpolation rule.

Правила интерполяция оценки наибольшего напряжение в конструкциях, подверженных ползучести

Арет В.А., Елисеев А.М., Федорова Л.А.
kingalya@mail.ru

Санкт-Петербургский национальный исследовательский университет ИТМО
Институт холода и биотехнологий
191002, Санкт-Петербург, ул. Ломоносова, 9

Несколько широко применяемых в пищевой промышленности простых конструкций, изготовленных из материала, для которого скорость деформации ползучести пропорциональна n-ой степени напряжения, рассматриваются с точки зрения определения направления, в котором наибольшее напряжение в конструкции изменяется в зависимости от показателя n, когда геометрия конструкции и возложенная на неё нагрузка остаются неизменными.

На основе изучения результатов предлагается простое линейное правило интерполирования наибольшего напряжения между простыми случаями n=1 и n=∞,

соответствующими точным решениям линейной упругости и идеальной пластичности материала. Правило интерполирования для рассматриваемых конструкций было проверено на основе К-метода [2].
Ключевые слова: ползучесть, относительный коэффициент концентрации напряжений; правило интерполирования

Introduction

In the design of mechanical engineering structures to operate in the creep range of the material of which they are made, it is often desirable to have an estimate of the greatest stress which will occur in the structure in the presence of creep. For a structure designed to operate over long periods of time, it is important that there should be an extended secondary creep phase at the highest stress level everywhere in the structure, while for a structure of shorter lifetime an estimate of the greatest stress may be useful in estimating the rupture life of the structure.

Unfortunately the stress-strain-time-temperature characteristics of engineering materials are usually so complicated that even if they could be expressed in terms of mathematical relationships the resulting structural calculations would be very complicated indeed.

In many cases results useful as a guide to the designer can be obtained by studying analytically the structural action corresponding to the secondary creep phase of the material. Analysis of this type has obvious relevance to structures designed for long-time operation, but, less obviously, it is also useful in the study of structures intended for shorter lifetimes, where primary creep plays an important role, if the results are used with engineering judgment [4].

Secondary creep analysis of structures

The secondary creep behavior of materials in uniaxial tension is specified by the equation

$$\dot{\varepsilon} = f(\sigma) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

for

$$\sigma < \sigma_t \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

where $\dot{\varepsilon}$ represents secondary creep rate,

σ represents stress

and *f(σ)* is a function of stress which characterizes the material at the appropriate temperature. The stress σ_t is a somewhat ill-defined limit below which the secondary creep phase exists for the proposed lifetime of the structure, and above it does not.

The simplest form of equation (1) is for *f(σ)* to be a single power term in σ:

$$\dot{\varepsilon} = B\sigma^n \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (3)$$

where *B* and *n* represent the material properties at the appropriate temperature. (It is understood as usual that a minus sign is to be inserted in equation (3) where otherwise $\dot{\varepsilon}$ and σ would be opposite sign)

After form of equation (1) which usually fits experimental data rather better than that equation (3) over wide ranges of stress levels is:

$$\dot{\varepsilon} = C\sinh\left(\frac{\sigma}{\sigma_0}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (4)$$

where *C* and σ_0 represent the material properties at the appropriate temperature. From the analytical point of view the simple law, equation (3), has much to recommend it in preference to equation (4), and other simple forms of equation (1), as it leads to neater analysis. An especially convenient property of equation (3) is that the special cases n=1 and n→∞ are exact analogues of linear elasticity and rigid-perfect plasticity respectively [1,2]. The fact that the real material behavior may be closer to equation (4) than to equation (3) fortunately is not a strong argument in favor of equation (4), especially in cases where the interest is in the most highly stressed parts of the structure; for equation (3), with suitably chosen values of *n* and *B*, closely approximates equation (4) over at least a decade of strain rate. There is always, of course, experimental scatter which makes determination of the constants *B* and *n* in equation (3) or of *C* and σ_0 in equation (4) rather imprecise.

This somewhat intuitive argument in favor of the simpler law, equation (3), is strengthened by the fact that results of analyses using equation (4) (such as are found in reference [2], for example) can usually be correlated extremely well on the basis of an *n*-power law.

Although structural analysis using the simple power law, equation (3), is less difficult than that using equation (4), it is unfortunately considerably more difficult (except in specially simple structures) for general values of *n* than for the two special cases *n=1* and *n*→∞; and values of *n* other than unity are the ones which are normally of interest, as they fit experimental data best for most engineering materials.

The establishment of interpolation schemes

The object of this paper is to investigate ways of estimating the value of greatest stress in a structure of specified geometry and carrying a giving load for any value of the exponent *n* in equation (3) without actually performing a non-linear analysis, by interpolating in some way between the answers found in the two specially simple cases, $n=1$ and $n\to\infty$. In other words, the object of the work is to establish simple ways of interpolating between solutions in linear elasticity and perfect plasticity for any value of the exponent *n*. In this investigation the energy dissipation rate interpolation scheme has been dropped in favor of a stress interpolation scheme which, is more generally applicable, and is almost always conservative.

In order to formulate a general interpolation scheme for the greatest stress in terms of *n*, it is necessary to know in general the manner in which the greatest stress does depend upon the value of exponent *n*. Even if general theorems in this field can be established, they are likely to be much more difficult to use to obtain close answers to problems than those that deal, for example, with effects integrated over the whole structure.

The only alternative therefore is to study the relationship between the greatest stress and the exponent *n* for a variety of structures. This paper describes stress analysis for six simple structures, and suggests a simple scheme of interpolation based on the results, which may well be valid in general.

Six structural types

The six simple structures were chosen to derive their structural action from a variety of primary structural effects:

(a) Bending in one dimension,
(b) Bending in two dimensions,
(c) Plane stress,
(d) Plane strain.

Details of the structures are given in Table 1. In general it seems reasonable to expect that the behavior of a structure and in particular the relationship between the greatest stress and the exponent *n* should depend upon the primary structural effect, to some extent independently of the complexity of the structure as a whole. Thus any generalizations which can be made from simple structures depending on different primary structural effects may well be valid for more complex structures depending on the same effects. In particular, the structures studied in this paper may be typical, as far as the dependence of the greatest stress upon the exponent *n* is concerned, of a very wide group of mechanical engineering structures.

Creep law for triaxial stress

For structures in which the material is in state of pure tension or compression throughout (which have, of course, only ideal existence) equation (3) is adequate to specify the material behavior, and ‘greatest stress’ means simple the largest absolute value of σ anywhere in the structure. For structures in which the material is subject to triaxial stress it is necessary to generalize equation (3) and to define an *effective* stress parameter for use in discussion of greatest stress, for it is obviously inadequate to think in terms of individual stress components.

The usual choice of a simple and rigorous generalization of equation (3) to triaxial stress lies between generalizations corresponding, in the limit $n \to \alpha$, to the von Mises or the Tresca yield condition respectively.

For some problems the Tresca-type idealization of real material is advantageous as it leads to simple solutions and gives good estimates of the over-ali energy-dissipating characteristics of the structure. However, for problems in which the *details* of stress distribution are to be studied it seems more satisfactory to use a material idealization in which the strain rate tensor is uniquely determined by the stress tensor. A simple, convenient, and physically realistic law is [3]

$$\dot{\varepsilon}_{ij} = B(3J_2)^{\frac{n-1}{2}}\left(\frac{3S_{ij}}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where S_{ij} is the stress deviation and J_2 is the second invariant of the stress deviation; $2J_2 = S_{ij} * S_{ij}$. For $n=1$ equation (5) describes a material analogous to a linear elastic material with Poisson’s ratio equal to ½. For the remainder of this paper use of the term ‘linear elastic’ will imply this special value of Poisson’s ratio. In so far as the rate of dissipation of energy per unit volume has stress dependence involving only J_2,

$$\dot{\varepsilon}_{ij\,\sigma_{ij}} = B(3J_2)^{(n+1)/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

it seems reasonable to regard the quantity

$$\overline{\sigma} = \sqrt{(3J_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

as ‘stress’ for the purposes of discussing greatest stress. In the remainder of this paper the term ‘greatest stress’ will be taken to mean the greatest value of σ anywhere in the structure.

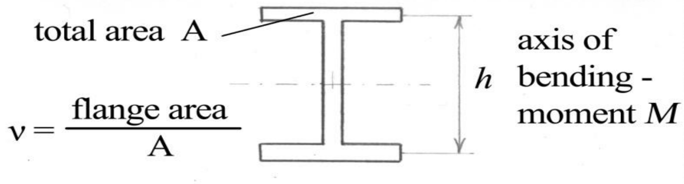
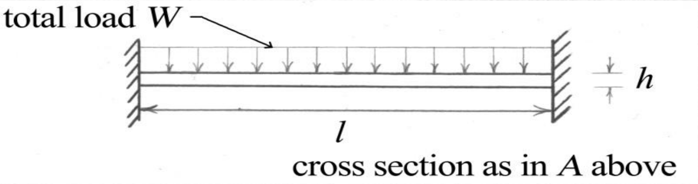
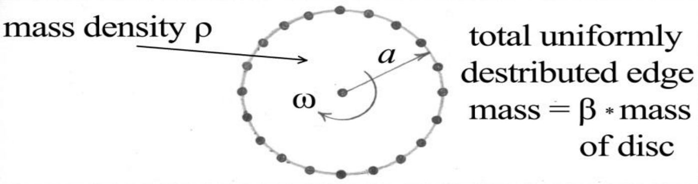
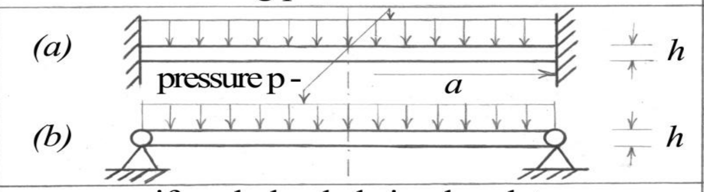
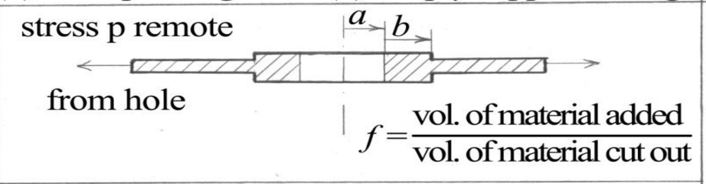
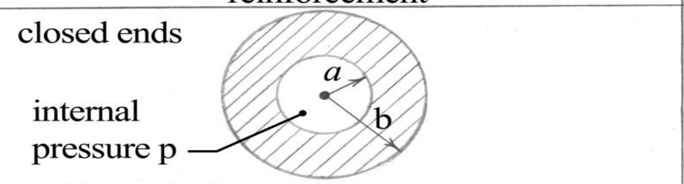
A	 I - section in pure bending	$\sigma_{max,\, n=1} = \frac{M}{Ah} \cdot \frac{6}{2\nu+1}$
B	 uniformly loaded beam with clamped ends	$\sigma_{max,\, n=1} = \frac{Wl}{Ah} \cdot \frac{1}{2\nu+1}$
C	 rotating parallel - sided disc	$\overline{\sigma}_{max,\, n=1} = \frac{\rho a^2 \omega^2}{g} \left(\frac{7}{16} + \frac{\beta}{2}\right)$
D	 uniformly loaded circular plate *(a) clamped edge (b) simply supported edge*	(a) $\overline{\sigma}_{max,\, n=1} = p\, \frac{a^2}{h^2} \cdot \frac{3\sqrt{3}}{6}$ (b) $\overline{\sigma}_{max,\, n=1} = p\, \frac{a^2}{h^2} \cdot \frac{21}{16}$
E	 symmetrical stretched sheet with hole and ring reinforcement	$\sigma_{max,\, n=1} = p\, \frac{2}{1 + \frac{3}{4} \cdot \frac{fa^2}{b^2}}$
F	 thick - walled tube under internal pressure	$\overline{\sigma}_{max,\, n=1} = p\, \frac{b^2\sqrt{3}}{b^2 - a^2}$

Table 1.

It has already been mentioned that in the case $n\to\infty$ the material is analogous to a rapid-perfectly plastic material (under monotonically increasing load). This may be seen if B in equation (3) is replaced by $\varepsilon_0 \sigma_0^n$ where σ_0 is the yield stress of the material. A little care is required in specifying what is meant by the stress distribution under a given set of loads as $n\to\infty$, because whereas for all finite values of n the structure can only deforms under any specified load, for $n \to \infty$ the structure can only deform if the applied load happens to correspond to the collapse load for the structure made of material having yield stress σ_0. We shall assume throughout this paper that in cases $n \to \infty$ the yield stress σ_0 is adjusted to magnitude so that the structure is in a collapse state under the application of the specified loads.

In structures C,D and E (Table 1) the material is in a state of bi-axial stress with principal directions determined by symmetry. In these conditions, putting $\sigma_3 = 0$ in equation (5) gives:

$$\dot{\varepsilon_1} = B\overline{\sigma}^{n-1}\left(\sigma_1 - \frac{1}{2}\sigma_2\right)$$

$$\dot{\varepsilon_2} = B\overline{\sigma}^{n-1}\left(\sigma_2 - \frac{1}{2}\sigma_1\right)$$

where

$$\overline{\sigma} = (\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2)^{1/2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

In structure F a three-dimensional state of stress exists, and equation (5) must be used. The problem is simplified, however, by the fact that by symmetry the axial strain rate is constant over the whole cross-section.

Choice of a reference state of stress

Stress concentration effects are usually discussed with respect to a reference state of stress. To some extent the definition of the reference stress is arbitrary. Where in a structure the bulk of the material is stressed to approximately the same level it is appropriate to use this bulk stress as the reference stress. Examples are the problem of a stretched sheet with a hole (structure E) and (in the field of shells) the cylindrical or spherical pressure vessel with a small nozzle or other geometrical discontinuity.

For most of the structures considered in this paper there is no such obvious reference state of stress, so some device is necessary if comparative stress concentration effects are to be studied. For the purposes of this paper it seems reasonable to use as the reference state of stress the greatest stress in the structure for the case *n=1*.

In general, as *n=1* defines an especially simple material, the greatest stress may be determined without too much difficulty (by linear-elastic analysis) in terms of the applied load and the geometrical parameters of the structure.

We thus define a ‘Relative Stress Concentration Factor’, F_r by:

$$F_r = \frac{\mathit{Greatest\ stress\ in\ structure\ for\ material}\ n = r}{\mathit{Greatest\ stress\ in\ structure\ for\ material}\ n = 1} \quad . \; . \; . \; . \quad (9)$$

it being understood that the geometry of the structure and the load carried are the same for the cases *n=1* and *n=r*.

Details of the solutions

The structures considered are shown schematically in Table 1.

All the usual assumptions of linear-elastic small deflection theory are used; thus we learn nothing, for example, about *local* stress concentration effects in the regions where plates join thicker plates (structure E) or rigid foundations (structure D). Nor do we learn anything about plates with membrane action.

Thus while it may seem paradoxical that acute, local, stress concentration effects have been ignored, it seems clear that the study, in so far as it leads to empirical general conclusions, may be of some use in tackling problems of local stress concentration factors.

In Fig.1 the relative stress concentration factor F_m is plotted against the material parameter m, which is the reciprocal of the exponent n:

$$m = \frac{1}{n}$$

The factor F_m is defined as in equation (9) but with an obvious change of subscript.

For most of the structures solutions were obtained, in addition to those for $n=1$ and $n\to\infty$ ($m=1$ and $m=0$, respectively) for $m = 0.1, 0.2, 0.4, 0.6,$ *and* 0.8.

Table 1 gives for each structure the value of the greatest stress, $\overline{\sigma}_{\max}$ in terms of the load and geometrical parameters for the case $n=1$. Use of this expression in conjunction with the appropriate graph in Fig.1 gives the value of $\overline{\sigma}_{\max}$ for any load and any value of the exponent n.

Schemes for interpolation

Consider now the problem of interpolation for F_m. Let us suppose that we have performed a linear-elastic analysis (with Poisson's ratio equal to ½) and a perfectly plastic analysis of the structure, and have in consequence obtained values of $\sigma_{\max,\, n=1}$ and F_0. We wish to interpolate between unity and F_0 for F_m and hence, using equation (9) to obtain an estimate of the greatest stress for the appropriate value of m. Clearly, from Fig.1 a simple linear interpolation

$$F_m = F_0(1-m) + m \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

is unsatisfactory in general, for although it gives good results in some cases it would lead to serious underestimates of the greatest stress in others.

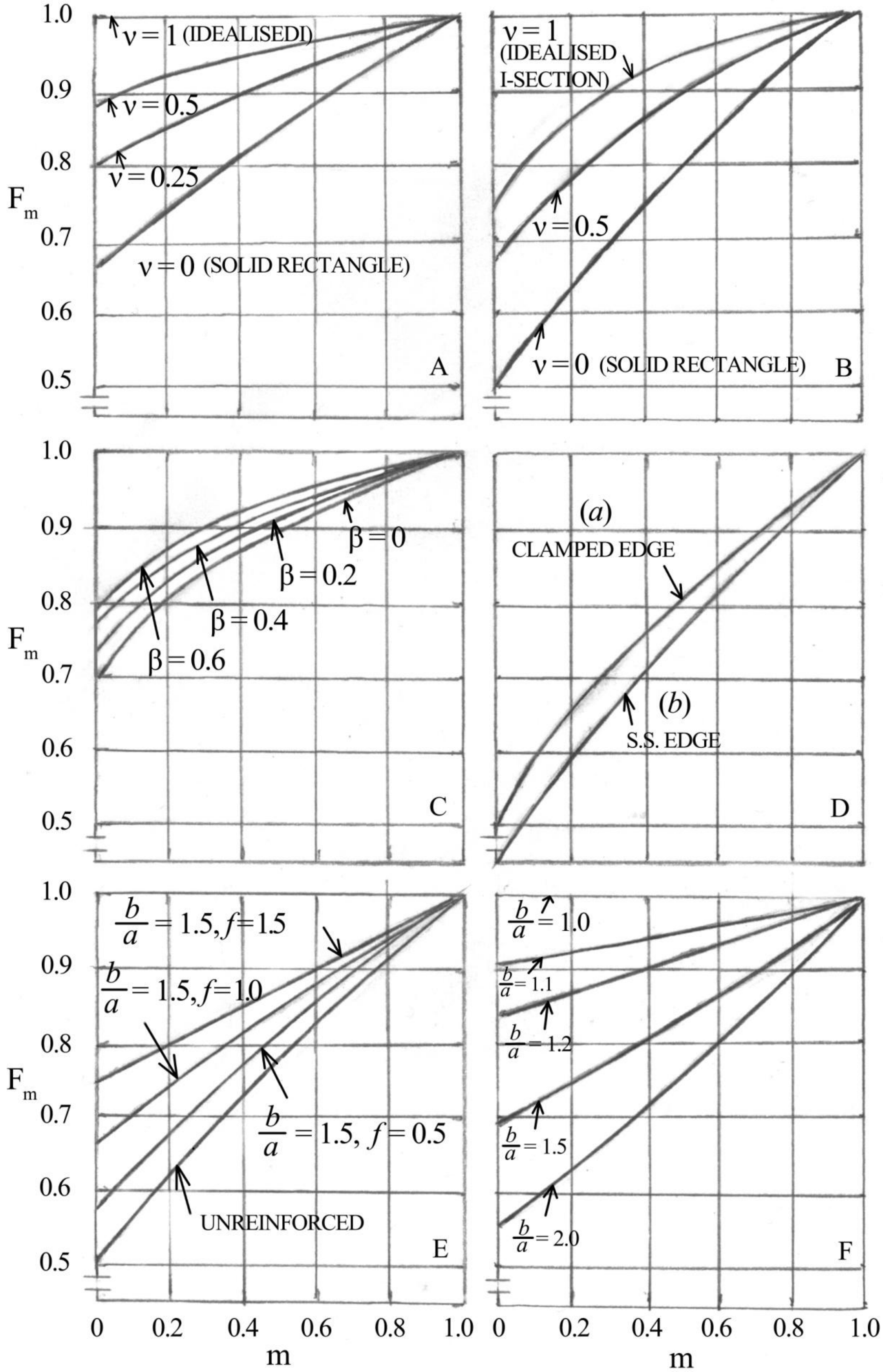


Fig.1 Results for structures shown in Table 1

It will be appreciated that an underestimate of the greatest stress could have serious consequences, especially if the greatest strain were a critical factor, as it might be for many structures required to operate for a specified period of time: see equation (3), where n may be large.

Clearly it would be useful to have, in addition to the values of $\sigma_{max,\ n=1}$ and F_0, an estimate of the curvature of the graph F_m , m. As will be shown later such an estimate may be obtained by performing an additional linear elastic 'thermal' stress type analysis.

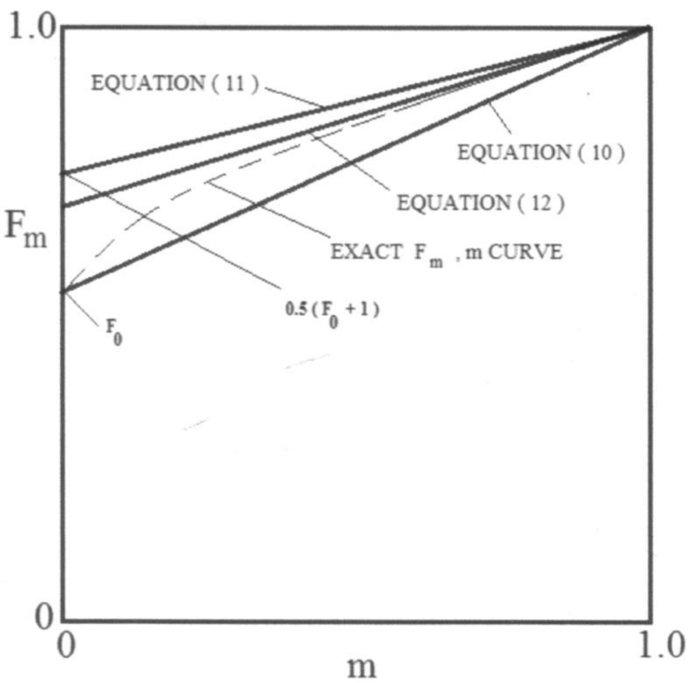


Fig.2 Interpolation curve, schematic

However, an analysis of this sort is by no means trivial, and it may well be desirable to make an estimate of the maximum stress solely on a basis of the linear elastic and perfectly plastic solutions.

In this case the following formula may be used:

$$F_m = F_0\left(\frac{1-m}{2}\right) + \left(\frac{1+m}{2}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (11)$$

Fig.2 shows equation (11) schematically, together with equation (10).

Judging by the results shown in Fig.1 equation (11) is almost always conservative: among the present examples it is only on the unsafe side for structure B, $\gamma = 1$ for $1 \le n \le 2.5$ and for structure C, $\beta = 0.6$ and 0.4 for $1 \le n \le 1.5$. The first of these exceptional cases, an idealized I-section beam with rigidly clamped ends is somewhat unrealistic; practical I-beams have γ equal approximately to ½ , for which equation (11) is just on the safe side for all values of *n*. the second case is perhaps more serious, for it means that equation (11) becomes less conservative as the peripheral mass increases. However, equation (11) is in the safe side for values of $n>3$ and is only slightly on the unsafe side in any case.

In the structures for which the F_m , m relationship is nearly linear, equation (11) tends to be very conservative, particularly for the lower values of *m* (high values of *n*); but is general this degree of conservatism should be tolerable, in view of many other uncertainties.

The analysis analogous to that for thermal stress, mentioned above analogous gives the value of the slope dF_m/dm at $m=1$. Call this slope λ. The line

$$F_m = 1 - \lambda(1 - \mathrm{m}) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

is thus the tangent to the curve F_m , m at $m=1$ (see Fig.2) and the difference in slope between it and line (10) is thus a measure of the curvature of the curve F_m ,m, assuming (see above) that there are no points of equation. If line (12) lies below line (10) it seems safe to use equation (10) for interpolation purposes, as the curve F_m , m is concave upwards over the whole range of *n*. If line (12) lies above line (10), equation (12) probably provides a safe interpolation scheme in which the degree of conservatism increases as *n* increases. If line (12) lies above line (11) this would seem to imply that the judgment which produced the constants in equation (11) was not sufficiently cautious. If line (12) is only slightly above line (11) equation (11) may well be conservative for, say, $n >2$, and acceptably unconservative for $n < 2$.

Conclusion

The simple interpolation procedures presented in this paper is intended to provide a useful extra analytical and design tool for the engineer. Application of the rules does not of course in any way absolve the engineer of the responsibility of avoiding high local stress concentrations by suitable detail design, etc., or of paying attention to the determination for loads. A particularly important design consideration for structures operating in the creep range is the distribution of temperature over the structure: this has not been considered at all in this work.

Estimate of the greatest stress for the structures to be considered was made according to *K*-method [2]. The values to be obtained are close to those determined on the base interpolation rule. They are not given here because their inclusion would made this paper too long.

In spite of these obvious shortcomings it is hoped that the results of this paper may remove a little of the design guesswork in a difficult design field.

References:

1. Malinin N.N. Applied theory of plasticity and creep. Moscow: "Mashinostroenie", 1975.
2. Kochanov L.M. Creep theory. Moscow: "Mashinostroenie", 1973.
3. Rabotnov Yu. N. Creep rupture // Proceedings applied mechanics conference. Stanford University, 1968. P. 342-349
4. Murakami S., Imazumi T. Mechanical description of creep damage state and experimental verification // J. mec. theor. et appl. 1982. Vol.1. №5. P. 743-761

Aret V.A. и др. Interpolation rule for estimating the greatest stress in a structure subject to creep/ V.A. Aret, A.M. Eliseev, L.A. Fedorova // Научный журнал НИУ ИТМО, 2013. - №3. [Электронный ресурс]: http://www.processes.ihbt.ifmo.ru